An Enriched Small Object Argument Over a Cofibrantly Generated Base

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Jan Jurka
jurka@math.muni.cz

Department of Mathematics and Statistics, Faculty of Science, Masaryk University

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Outline

- Quillen [**Qui67**] introduced a transfinite construction of weak factorization systems, dubbed the *small object argument*.
- Our main goal is to introduce an enriched version of the small object argument.
- Along the way, we introduce a variation of the Day convolution in which we use copowers instead of the monoidal product.

Small Object Argument

First, the classical small object argument in a category $\mathcal{K}$.

Definition.

a morphism $f\colon A \to B$ is said to have the *left lifting property* with respect to a morphism $k\colon C \to D$, which we denote $f \boxslash k$, if for each commutative square

$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[s]{} & D \end{array}$$

there exists a diagonal $d\colon B \to C$ making the two triangles below commute.

$$\begin{array}{ccc} A & \xrightarrow{r} & C \\ {\scriptstyle f}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle k} \\ B & \xrightarrow[s]{} & D \end{array}$$

Given a class $\mathcal{J}$ of morphisms in $\mathcal{K}$:

$$\mathcal{J}^{\square} := \{k \mid \forall f \in \mathcal{J} : f \,\square\, k\}$$

$$^{\square}\mathcal{J} := \{f \mid \forall k \in \mathcal{J} : f \,\square\, k\}$$

Definition.

A *weak prefactorization system* $\mathscr{F} = (\mathscr{L}, \mathscr{R})$ on $\mathcal{K}$ is a pair of classes of morphisms in $\mathcal{K}$ such that $\mathscr{L}^{\square} = \mathscr{R}$ and $\mathscr{L} = {}^{\square}\mathscr{R}$.
Furthermore, a weak prefactorization system $\mathscr{F}$ is called *a weak factorization system* if for each morphism f there exists a pair of morphisms $g \in \mathscr{L}$, $h \in \mathscr{R}$ such that $f = h \cdot g$.
Moreover, a weak factorization system $\mathscr{F}$ is said to be *cofibrantly generated* if there exists a set $\mathcal{J}$ of morphisms such that $\mathscr{R} = \mathcal{J}^{\square}$.

Small Object Argument.

Suppose that $\mathcal{K}$ is a category with pushouts and transfinite composites, and $\mathcal{I}$ a **set** of morphisms in $\mathcal{K}$ whose domains are presentable. Then for each morphism f in $\mathcal{K}$ there exists a factorization $f = m \cdot e$ such that $e \in {}^{\square}(\mathcal{I}^{\square})$ and $m \in \mathcal{I}^{\square}$.

Corollary.

Under the same assumptions, we obtain a weak factorization system $({}^{\square}(\mathcal{I}^{\square}), \mathcal{I}^{\square})$ on $\mathcal{K}$.

The construction takes suitable pushouts of morphisms in $\mathcal{I}$ and then takes a (nested) transfinite composite of these pushouts.

An important ingredient is the stability of classes ${}^{\square}\mathcal{I}$ under pushouts and transfinite composites.

The aim of this talk is to introduce an enriched version.

Interlude: Copowers in Categories of $\mathcal{V}$-Functors

Setting.

- $\mathcal{V}$ is a *cosmos*, i.e. a bicomplete symmetric monoidal closed category,
- $\mathcal{K}$ is a copowered $\mathcal{V}$-category admitting coends of the form (1),
- $\mathcal{A}$ is a small monoidal $\mathcal{V}$-category.

Theorem.

(i) The $\mathcal{V}$-category $[\mathcal{A}, \mathcal{K}]$ is a left $[\mathcal{A}, \mathcal{V}]$-actegory such that the action on a fixed object of $[\mathcal{A}, \mathcal{K}]$ always has a right adjoint, and

(ii) the category $[\mathcal{A}, \mathcal{K}]_0$ is a copowered $[\mathcal{A}, \mathcal{V}]$-category.

Notation.

By $\odot$ we denote copowering in $\mathcal{K}$ and also the monoidal product in $\mathcal{V}$. The monoidal product in $\mathcal{A}$ is denoted by m.

Definition.

Given $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{V}$, $X\colon \mathcal{A} \to \mathcal{K}$, we define the $\mathcal{V}$-functor $F * X\colon \mathcal{A} \to \mathcal{K}$ by:

$$(F * X)(x) := \int^{a,b \in \mathcal{A}} \big(\mathcal{A}(m(a,b),x) \odot F(a)\big) \odot X(b), \tag{1}$$

Remark.

The definition above is the copowering/action in the previous theorem. Also, note that it is just like the Day convolution except that we use copowering in $\mathcal{K}$ in the second occurrence of $\odot$.

The associativity of the action $*$ will be essential for our enriched small object argument.

The internal hom in $[\mathcal{A}, \mathcal{K}]_0$ is given by:

$$\langle X, Y \rangle := \int_{a \in \mathcal{A}} \mathcal{K}\big(X(a), Y(m(-, a))\big),$$

i.e.

$$[\mathcal{A}, \mathcal{K}](F * X, Y) \cong [\mathcal{A}, \mathcal{V}](F, \langle X, Y \rangle).$$

When $\mathcal{V} = \mathbf{Set}$, the enrichment of $[\mathcal{A}, \mathcal{K}]_0$ over $[\mathcal{A}, \mathbf{Set}]$ has been described directly by McDermott and Uustalu [**MU22**]. They did it in the language of locally $\mathcal{A}$-graded categories, which are an elementary formulation of $[\mathcal{A}, \mathbf{Set}]$-enriched categories due to Wood [**Woo76**].

The Enriched Category of Arrows

We specialize the preceding to $\mathcal{A} = \mathbf{2}$, where $\mathbf{2}$ is the free $\mathcal{V}$-category on the category with two objects $0, 1$ and a single non-identity morphism $0 \to 1$. Monoidal product on $\mathbf{2}$ is given by the cartesian product, on objects as follows $m(x, y) := \min(x, y)$.

Then $[\mathbf{2}, \mathcal{V}]$ and $[\mathbf{2}, \mathcal{K}]$ are the $\mathcal{V}$-categories of morphisms in $\mathcal{V}_0$ and in $\mathcal{K}_0$, respectively.

Hom-objects in $[\mathbf{2}, \mathcal{K}]$ as a $\mathcal{V}$-category are given by $\mathrm{Sq}(f, k)$ from the following definition.

Definition.

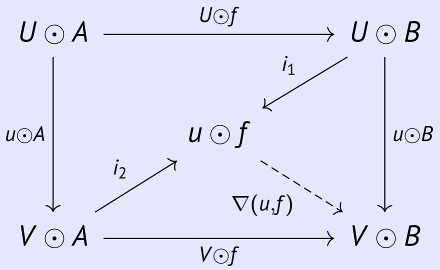
For each pair of morphisms $f\colon A \to B$, $k\colon C \to D$ in $\mathcal{K}_0$, define $\mathrm{Sq}(f, k)$ to be the *object of squares connecting f to k*, i.e. the pullback-object in the following pullback square.

$$\begin{array}{ccc} \mathrm{Sq}(f,k) & \xrightarrow{p_1} & \mathcal{K}(A,C) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \mathcal{K}(A,k)} \\ \mathcal{K}(B,D) & \xrightarrow[\mathcal{K}(f,D)]{} & \mathcal{K}(A,D) \end{array}$$

The existence of coends of the form (1) is equivalent to the existence of pushouts of the form $u \odot f$ from the following definition. Note that ∇ is the action $*$ from previous slides.

Definition.

Suppose that u is a morphism in $\mathcal{V}_0$ and $f\colon A \to B$ is a morphism in $\mathcal{K}_0$. Then the *pushout-product* $\nabla(u,f)\colon u \odot f \to V \odot B$ is the induced morphism depicted in the following diagram, where (i_1, i_2) is a pushout of $U \odot f$ and $u \odot A$.



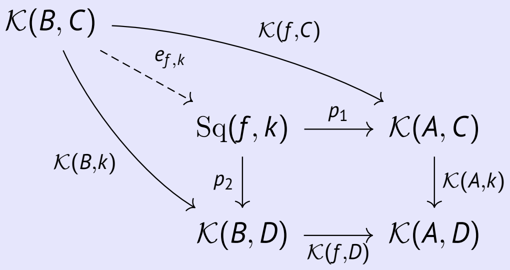
Definition.

We say that a class $\mathscr{S}$ of morphisms in $\mathcal{V}_0$ is *stable under corners* if whenever $u, v \in \mathscr{S}$, then $\nabla(u, v) \in \mathscr{S}$.

The hom-object $\langle f, k\rangle$ for $[\mathbf{2}, \mathcal{K}]$ as a $[\mathbf{2}, \mathcal{V}]$-enriched category becomes $e_{f,k}$ from the following definition.

Definition.

Suppose that $f : A \to B$, $k : C \to D$ are morphisms in $\mathcal{K}_0$. Then define $e_{f,k} : \mathcal{K}(B, C) \to \mathrm{Sq}(f, k)$ to be the induced morphism depicted below.



Enriched Lifting Properties

Setting.

- $\mathcal{V}$ is a cosmos,
- $\mathcal{K}$ is a copowered $\mathcal{V}$-category that has pushouts and transfinite composites,
- $\mathscr{F} = (\mathscr{L}, \mathscr{R})$ is a weak prefactorization system on $\mathcal{V}_0$.

Definition.

Suppose that $f : A \to B$, $k : C \to D$ are morphisms in $\mathcal{K}_0$. Then we write $f \overset{\mathscr{F}}{\square} k$ if $e_{f,k} \in \mathscr{R}$, where $e_{f,k} : \mathcal{K}(B, C) \to \mathrm{Sq}(f, k)$ is the induced morphism from the previous slides.

Furthermore, if $f \overset{\mathscr{F}}{\square} k$, then we say that f has the *left $\mathscr{F}$-lifting property* with respect to k, or equivalently that k has the *right $\mathscr{F}$-lifting property* with respect to f. Moreover, if $\mathcal{I}$ is a class of morphisms in $\mathcal{K}_0$, then we define

$$\mathcal{I}^{\overset{\mathscr{F}}{\square}} := \{k \in [\mathbf{2}, \mathcal{K}] \mid \forall f \in \mathcal{I} : f \overset{\mathscr{F}}{\square} k\},$$

$$^{\overset{\mathscr{F}}{\square}}\mathcal{I} := \{f \in [\mathbf{2}, \mathcal{K}] \mid \forall k \in \mathcal{I} : f \overset{\mathscr{F}}{\square} k\}.$$

Examples.

- In the case $\mathcal{V} = \mathbf{Set}$, $\mathscr{F} =$ (injective, surjective) we capture the ordinary (weak) lifting property, i.e. $f \overset{\mathscr{F}}{\square} k$ iff $f \square k$.
- In the case $\mathcal{V} = \mathbf{Set}$, $\mathscr{F} =$ (all functions, bijections) we capture the strong lifting property in which the diagonal is required to be unique.
- Suppose that $\mathcal{V} = \mathbf{Cat}$,

  $\mathscr{F} =$ (injective on objects, surjective equivalences). Then $f \overset{\mathscr{F}}{\square} k$ iff $e_{f,k}$ is a surjective equivalence, which happens iff for each pair of 1-cells $r \colon A \to C$, $s \colon B \to D$ satisfying $k \cdot r = s \cdot f$ there exists a diagonal $d \colon B \to C$ such that $d \cdot f = r$, $k \cdot d = s$, and furthermore if $d, d' \colon B \to C$ are 1-cells, and $\theta \colon d \cdot f \Rightarrow d' \cdot f$, $\theta' \colon k \cdot d \Rightarrow k \cdot d'$ are 2-cells such that $k * \theta = \theta' * f$, then there exists a unique 2-cell $\varphi \colon d \Rightarrow d'$ such that $\varphi * f = \theta$ and $k * \varphi = \theta'$.

Definition.

An *enriched weak $\mathscr{F}$-factorization system* on $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{K}_0$ such that each morphism $h\colon A \to B$ in $\mathcal{K}_0$ has a factorization $h = g \cdot f$ such that $f \in \mathcal{L}$, $g \in \mathcal{R}$, and furthermore $\mathcal{L} = {}^{\overset{\mathscr{F}}{\square}}\mathcal{R}$, $\mathcal{R} = \mathcal{L}^{\overset{\mathscr{F}}{\square}}$.

Suppose that $\mathcal{I}$ is a class of morphisms in $\mathcal{K}_0$.

Proposition.

The class $\overset{\mathscr{F}}{\square}\mathcal{I}$ is stable under pushouts in $\mathcal{K}$.

Proposition.

The class $\overset{\mathscr{F}}{\square}\mathcal{I}$ is stable under transfinite compositions in $\mathcal{K}$.

Proposition.

Suppose that $\mathscr{L}$ is stable under corners and admits pushouts of the form $u \odot f$. Then $\overset{\mathscr{F}}{\square}\mathcal{I}$ is stable under **copowered pushouts** in $\mathcal{K}$.

Recall that a *colimit in* $\mathcal{K}$ means a colimit in $\mathcal{K}_0$ that is sent by each representable functor $\mathcal{K}(-, K)\colon \mathcal{K}_0^{\mathrm{op}} \to \mathcal{V}_0$ to a limit in $\mathcal{V}_0$.

Here's what the stability under copowered pushouts means:

Proposition.

Suppose that $\mathscr{L}$ is stable under corners and admits pushouts of the form $u \odot f$. Moreover, suppose that we have a span

$$\begin{array}{ccc} u \odot f & \xrightarrow{(g,h)} & K \\ {\scriptstyle \nabla(u,f)}\downarrow & & \\ V \odot B & & \end{array}$$

in $\mathcal{K}_0$, where $u \colon U \to V$ is in $\mathscr{L}$, $f \colon A \to B$, $g \colon V \odot A \to K$, and $h \colon U \odot B \to K$ are morphisms in $\mathcal{K}_0$, and that the following square is a pushout in $\mathcal{K}$.

$$\begin{array}{ccc} u \odot f & \xrightarrow{(g,h)} & K \\ {\scriptstyle \nabla(u,f)}\downarrow & & \downarrow{\scriptstyle f'} \\ V \odot B & \xrightarrow[g']{} & L \end{array}$$

If $f \in {}^{\overset{\mathscr{F}}{\square}}\mathcal{I}$, then $f' \in {}^{\overset{\mathscr{F}}{\square}}\mathcal{I}$.

An important ingredient for our proof of the stability under copowered pushouts is the associativity of ∇ (i.e. of the variation $*$ of the Day convolution):

$$\nabla(v, \nabla(u, f)) \cong \nabla(\nabla(v, u), f),$$

where u, v are morphisms in $\mathcal{V}_0$ and f is a morphism in $\mathcal{K}_0$.

Enriched Small Object Argument

Remark.

We will prove our small object argument in the special case when $\mathscr{F}$ is a cofibrantly generated weak factorization system.

Definition.

Suppose that λ is a regular cardinal. An object K of $\mathcal{K}$ is said to be λ*-presentable in the enriched sense* [**Kel82**] if the functor $\mathcal{K}(K, -)\colon \mathcal{K}_0 \to \mathcal{V}_0$ preserves λ-directed colimits in $\mathcal{K}$. Moreover, an object K of $\mathcal{K}$ is said to be *presentable in the enriched sense* if there exists a regular cardinal μ such that K is μ-presentable.

Theorem.

Suppose that $\mathcal{V}$ is a cosmos, $\mathcal{K}$ is a copowered $\mathcal{V}$-category that has pushouts and transfinite composites, $\mathcal{I}$ is a **set** of morphisms in $\mathcal{K}_0$, $\mathcal{J}$ is a **set** of morphisms in $\mathcal{V}_0$, $\mathscr{F} = (\mathscr{L}, \mathscr{R})$ is a weak factorization system on $\mathcal{V}_0$ that's cofibrantly generated by $\mathcal{J}$, and

(i) the domains and codomains of morphisms in $\mathcal{I}$ are presentable in the enriched sense.

(ii) the domains and codomains of morphisms in $\mathcal{J}$ are presentable (in the unenriched sense), and

(iii) $\mathscr{L}$ is stable under corners.

Then for each morphism $f\colon K \to L$ in $\mathcal{K}_0$ there exists a factorization $f = m \cdot e$ such that e and m are morphisms in $\mathcal{K}_0$ satisfying $e \in {}^{\overset{\mathscr{F}}{\square}}(\mathcal{I}^{\overset{\mathscr{F}}{\square}})$ and $m \in \mathcal{I}^{\overset{\mathscr{F}}{\square}}$.

Corollary.

Under the same assumptions, we obtain an enriched weak $\mathscr{F}$-factorization system $({}^{\overset{\mathscr{F}}{\square}}(\mathcal{I}^{\overset{\mathscr{F}}{\square}}), \mathcal{I}^{\overset{\mathscr{F}}{\square}})$.

The construction takes suitable copowered pushouts of morphisms in $\mathcal{I}$ and then takes a (nested) transfinite composite of these copowered pushouts. In each step we use $|\mathcal{J}|$-many kinds of copowered pushouts and this is done by cyclings through $\mathcal{J}$.

$\mathcal{V} = \mathbf{Set}$

- $\mathcal{J} = \{\emptyset \to 1\}$, i.e. $\mathscr{F} = (\text{injective}, \text{surjective})$: Classical 1-categorical Small Object Argument for weak factorization systems. One kind of copowered pushouts.
- $\mathcal{J} = \{\emptyset \to 1, 2 \to 1\}$, i.e. $\mathscr{F} = (\text{all functions}, \text{bijections})$: 1-categorical Small Object Argument for orthogonal factorization systems. Two kinds of copowered pushouts.

$\mathcal{V} = \mathbf{Cat}$

$\mathcal{J}$ consists of the following three morphisms:

$$\emptyset \longrightarrow \bullet$$

$$(\bullet \quad \bullet) \longrightarrow (\bullet \to \bullet)$$

$$(\bullet \rightrightarrows \bullet) \longrightarrow (\bullet \to \bullet)$$

i.e. $\mathscr{F} =$ (injective on objects, surjective equivalences). This leads to a variant of a 2-categorical Small Object Argument. The construction uses three kinds of copowered pushouts.

$\mathcal{V} = \mathbf{Ch}$

$$\mathcal{J} = \{S^{n-1} \hookrightarrow D^n \mid n \in \mathbb{Z}\},$$

i.e. $\mathscr{R}$ are precisely the surjective quasi-isomorphisms. This leads to a variant of a dg-categorical Small Object Argument. The construction uses $\aleph_0$ kinds of copowered pushouts.

arXiv Preprint

More details can be found in the preprint:

An Enriched Small Object Argument Over a Cofibrantly Generated Base (arXiv:2401.05974)

Bibliography

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Thank you for your attention!

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